# SATURATED O-MINIMAL EXPANSIONS OF REAL CLOSED FIELDS

ANNALISA CONVERSANO, PAOLA D'AQUINO, AND SALMA KUHLMANN

Abstract. In [KKMZ02] the authors gave a valuation theoretic characterization for a real closed field to be $\kappa$-saturated, for a cardinal $\kappa \geq \aleph_0$. In this paper, we generalize the result, giving necessary and sufficient conditions for certain o-minimal expansion of a real closed field to be $\kappa$-saturated.

## 1. Introduction

A totally ordered structure $\mathcal{M} = \langle M, <, \dots \rangle$ (in a countable first order language containing $<$) is o-minimal if every subset of it which is definable with parameters in $M$ is a finite union of intervals in $M$. These structures have many interesting features. We focus here on the following: For $\alpha > 0$, $\mathcal{M}$ is $\aleph_\alpha$-saturated if and only if the underlying order $\langle M, < \rangle$ is $\aleph_\alpha$-saturated as a linearly ordered set ([AK94]). If $\mathcal{M}$ is an o-minimal expansion of a divisible ordered abelian group (DOAG), then $\langle M, < \rangle$ is a dense linear order without endpoints (DLOWEP). Now, $\aleph_\alpha$-saturated DLOWEP are well understood, they are Hausdorff's $\eta_\alpha$ - sets, see [R]. The above equivalence provides therefore a characterization of $\aleph_\alpha$-saturation of such o-minimal expansions for $\alpha \neq 0$. We are reduced to characterising $\aleph_0$-saturation. This problem was solved in [Ku90] and in [KKMZ02] for DOAG and for real closed fields, respectively.

In this paper we generalize this result to power bounded o-minimal expansions of real closed fields, see Theorem 5.2. Miller in [M1] proved a dichotomy theorem for o-minimal expansions of the real ordered field by showing that for any o-minimal expansion $\mathcal{R}$ of $\mathbb{R}$ not polynomially bounded the exponential function is definable in $\mathcal{R}$. Later, Miller extended this result to any o-minimal expansion of a real closed field (see [M2]) by replacing *polynomially bounded* by *power bounded.*

In [DKS10] it was shown that a countable real closed field is recursively saturated if and only if it has an integer part which is a model

*Date*: May 17, 2012.

2000 *Mathematics Subject Classification.* Primary: 06A05, 12J10, 12J15, 12L12, 13A18; Secondary: 03C60, 12F05, 12F10, 12F20.

*Key words and phrases.* natural valuation, value group, residue field, pseudo-Cauchy sequences, o-minimal expansion of a real closed field, definable closure, dimension, $\aleph_0$-saturation.

of Peano Arithmetic (see [DKS10] for these notions). In a forthcoming paper, we give a valuation theoretic characterization of recursively saturated real closed fields (of arbitrary cardinality), and their o-minimal expansions.

## 2. Background on o-minimal structures

We recall some properties of o-minimal structures. Let $\mathcal{L}$ be a countable language containing $<$, and let $\mathcal{M} = \langle M, <, \dots \rangle$ be an o-minimal $\mathcal{L}$-structure. If $A \subset M$ then the algebraic closure $\operatorname{acl}(A)$ of $A$ is the union of the finite $A$-definable sets, and the definable closure $\operatorname{dcl}(A)$ is the union of the $A$-definable singletons. In general, $\operatorname{dcl}(A) \subseteq \operatorname{acl}(A)$, but in an o-minimal structure $\mathcal{M}$ they coincide. For example, if $\mathcal{M}$ is a divisible abelian group and $A \in M$ then the definable closure of $A$ coincides with the $\mathbb{Q}$ vector space generated by $A$, $\operatorname{dcl}(A) = \langle A \rangle_{\mathbb{Q}}$. If $\mathcal{M}$ is a real closed field then the definable closure of $A \subset M$ is the relative real closure of the field $\mathbb{Q}(A)$ in $M$, i.e. $\operatorname{dcl}(A) = \mathbb{Q}(A)^{rc}$.

Notice that over a countable language $\mathcal{L}$ the cardinality of the definable closure of a set $A$ is:

$$|\operatorname{dcl}(A)| = \begin{cases} \aleph_0 & \text{if} \quad |A| \leq \aleph_0 \\ |A| & \text{if} \quad |A| > \aleph_0 \end{cases} \tag{1}$$

In [PS] it is proved that in any o-minimal structure $\mathcal{M}$ the operator *dcl* is a pregeometry, i.e. it satisfies the following properties:

(1) for any $A \subseteq M$, $A \subseteq \operatorname{dcl}(A)$;
(2) for any $A \subseteq M$, $\operatorname{dcl}(A) \subseteq \operatorname{dcl}(\operatorname{dcl}(A))$;
(3) for any $A \subseteq M$, $\operatorname{dcl}(A) = \bigcup\{\operatorname{dcl}(F) : F \subseteq A, \quad F \text{ finite}\}$
(4) *(Exchange Principle)* for any $A \subseteq M$, $a, b \in M$ if $a \in \operatorname{dcl}(A \cup \{b\}) - \operatorname{dcl}(A)$ then $b \in \operatorname{dcl}(A \cup \{a\})$.

The Exchange Principle guarantees that in any o-minimal structure $\mathcal{M}$ there is a good notion of independence:

A subset $A \subset M$ is *independent* if for all $a \in A$, $a \notin \operatorname{dcl}(A - \{a\})$. If $B \subset M$ we say that $A$ is *independent over* $B$ if $a \notin \operatorname{dcl}(B \cup (A - \{a\}))$. A subset $A \subseteq M$ is said to generate $\mathcal{M}$ if $M = \operatorname{dcl}(A)$. An independent set $A$ that generates $\mathcal{M}$ is called a basis. The Exchange Principle guarantees that any independent subset of $M$ can be extended to a basis, and all basis for $\mathcal{M}$ have the same cardinality. So a basis for $\mathcal{M}$ is any maximal independent subset. The *dimension* of $\mathcal{M}$ is the cardinality of any basis. It is easy to extend the notion of a basis of $\mathcal{M}'$ over $\mathcal{M}$ when $\mathcal{M} \preceq \mathcal{M}'$. Note that

$$\dim(\mathcal{M}') \leq |A| \tag{2}$$

We recall the notion of *prime* model of a theory $T$. Let $A \subseteq \mathcal{M} \models T$. The model $\mathcal{M}$ is said to be prime over $A$ if for any $\mathcal{M}' \models T$ with $A \subseteq M'$ there is an elementary mapping $f : \mathcal{M} \to \mathcal{M}'$ which is the

identity on $A$. For exampe, if $T$ is the theory of real closed fields the real closure of an ordered field $F$ is prime over $F$. It is well known, see [PS], that if $\mathcal{M}$ is an o-minimal structure, and $A \subseteq M$ then $Th(\mathcal{M})$ has a prime model over $A$, and this is unique up to $A$-isomorphism. For any subset $A \subseteq M$ it coincides with $\mathrm{dcl}(A)$. If $A = \emptyset$ then $\mathrm{dcl}(\emptyset) = P$ is the prime model of $T$.

Let us notice that if $\mathcal{M}$ is a real closed field, then the dimension of $\mathcal{M}$ over the prime field coincides with the transcendence degree of $\mathcal{M}$ over $\mathbb{Q}$.

## 3. $\aleph_\alpha$-SATURATED DIVISIBLE ORDERED ABELIAN GROUPS

We summarize the required background (see [Ku01] and [Ku90]). Let $(G, +, 0, <)$ be a divisible ordered abelian group. For any $x \in G$ let $|x| = \max\{x, -x\}$. For non-zero $x, y \in G$ we define $x \sim y$ if there exists $n \in \mathbb{N}$ such that $n|x| \geq |y|$ and $n|y| \geq |x|$. We write $x << y$ if $n|x| < |y|$ for all $n \in \mathbb{N}$. Clearly, $\sim$ is an equivalence relation. Let $\Gamma := G - \{0\}/\sim = \{[x] : x \in G - \{0\}\}$. We can define an order on $<_\Gamma$ in terms of $<<$ as follows, $[y] <_\Gamma [x]$ if $x << y$ (notice the reversed order).

**Fact 3.1.** *(a)* $\Gamma$ is a totally ordered set under $<_\Gamma$, and we will refer to it as the value set of $G$.
*(b)* The map

$$\begin{array}{rcl} v\colon G & \longrightarrow & \Gamma \cup \{\infty\} \\ 0 & \mapsto & \infty \\ x & \mapsto & [x] \quad (\text{if } x \neq 0) \end{array}$$

is a valuation on $G$ as a $\mathbb{Z}$-module, i.e. for every $x, y \in G$:
$v(x) = \infty$ if and only if $x = 0$, $v(nx) = v(x)$ for all $n \in \mathbb{Z}$, $n \neq 0$, and $v(x + y) \geq \min\{v(x), v(y)\}$.

($c$) For every $\gamma \in \Gamma$ the Archimedean component associated to $\gamma$ is the maximal Archimedean subgroup of $G$ containing some $x \in \gamma$. We denote it by $A_\gamma$. For each $\gamma$, $A_\gamma \subseteq (\mathbb{R}, +, 0, <)$.

**Definition 3.2.** Let $\lambda$ be an infinite ordinal. A sequence $(a_\rho)_{\rho<\lambda}$ contained in $G$ is said to be *pseudo Cauchy* (or *pseudo convergent*) if for every $\rho < \sigma < \tau$ we have

$$v(a_\sigma - a_\rho) \ < \ v(a_\tau - a_\sigma).$$

**Fact 3.3.** If $(a_\rho)_{\rho<\lambda}$ is pseudo Cauchy sequence then for all $\rho < \sigma$ we have

$$v(a_\sigma - a_\rho) = v(a_{\rho+1} - a_\rho).$$

**Definition 3.4.** Let $(a_\rho)_{\rho<\lambda}$ be a pseudo Cauchy sequence in $G$. We say that $x \in G$ is a *pseudo limit* of $S$ if

$$v(x - a_\rho) = v(a_\sigma - a_\rho) = v(a_{\rho+1} - a_\rho) \quad \text{for all } \rho < \sigma.$$

We now recall the characterization of $\aleph_\alpha$-saturation for divisible ordered abelian groups, see [Ku90].

**Theorem 3.5.** [Ku90] *Let $G$ be a divisible ordered abelian group, and let $\aleph_\alpha \geq \aleph_0$. Then $G$ is $\aleph_\alpha$-saturated in the language of ordered groups if and only*

1. *its value set is an $\eta_\alpha$-set*
2. *all its Archimedean components are isomorphic to $\mathbb{R}$*
3. *every pseudo Cauchy sequence in a divisible subgroup of value set $< \aleph_\alpha$ has a limit in $G$.*

Notice that in the case of $\aleph_0$-saturation the necessary and sufficient conditions reduce only to (1) and (2), see [Ku90].

## 4. $\aleph_\alpha$-saturated real closed fields

If $(R, +, \cdot, 0, 1, <)$ is an ordered field then it has a natural valuation $v$, that is the natural valuation associated to the ordered abelian group $(R, +, 0, <)$. We will denote by $G$ the value group of $R$ with respect to $v$, i.e. $G = v(R)$. If $(R, +, \cdot, 0, 1, <)$ is a real closed field then $G$ is divisible, and we will refer to the rational rank of $G$, $\mathrm{rk}(G)$, for the linear dimension of $G$ as a $\mathbb{Q}$-vector space.

For the natural valuation on $R$ we use the notations $\mathcal{O}_R = \{r \in R : v(r) \geq 0\}$ and $\mu_R = \{r \in R : v(r) > 0\}$, for the valuation ring and the valuation ideal, respectively. The residue field $k$ is the quotient $\mathcal{O}_R/\mu_R$, and we recall that it is a subfield of $\mathbb{R}$. Notice that in the case of ordered fields there is a unique archimedean component up to isomorphism, and if the field is real closed the archimedean component is the residue field.

A notion of pseudo Cauchy sequence is easily extended to any ordered field as in the case of ordered abelian groups.

The following characterization of $\aleph_\alpha$-saturated real closed fields was obtained in [KKMZ02].

**Theorem 4.1.** [KKMZ02, 6.2] *Let $R$ be a real closed field, $v$ its natural valuation, $G$ its value group and $k$ its residue field. Let $\aleph_\alpha \geq \aleph_0$. Then $R$ is $\aleph_\alpha$-saturated in the language of ordered fields if and only if*

1. *$G$ is $\aleph_\alpha$-saturated,*
2. *$k \cong \mathbb{R}$,*
3. *every pseudo Cauchy sequence in a subfield of absolute transcendence degree less than $\aleph_\alpha$ has a pseudo limit in $R$.*

In the proof of Theorem 4.1 the *dimension inequality* (see [P]) is crucially used in the case of $\aleph_0$-saturation. This says that the rational rank of the value group of a finite transcendental extension of a real closed field is bounded by the transcendence degree of the extension.

## 5. $\aleph_\alpha$-SATURATED EXPANSIONS OF A REAL CLOSED FIELD

We show now a generalization of Theorem 4.1 to o-minimal expansions of a real closed field $\mathcal{M} = (M, +, \cdot, 0, 1, <, \ldots)$.

The proof follows the lines of the previous characterizations. Also in this case some care is needed for $\aleph_0$-saturation. We need to bound the rational rank of the value group of a finite dimensional extension. (Recall from (1) that the cardinality of the definable closure of a finite set is infinite.) Analogues of the dimension inequality have been proved by Wilkie and van den Dries in more general cases.

Let $T$ be the theory of an o-minimal expansion of $\mathbb{R}$ and assume $T$ is *smooth*, see [W]. In [W] Wilkie showed that if $\mathcal{R}$ is a model $T$, and $\dim(\mathcal{R})$ is finite then $\mathrm{rk}(\mathcal{R}) \leq \dim(\mathcal{R})$. This result has been further generalized by van den Dries in [vdD] to *power bounded* o-minimal expansions of a real closed field. We recall that $\mathcal{M}$ is *power bounded* if for each definable function $f : \mathcal{M} \to \mathcal{M}$ there is $\lambda \in M$ such that $|f(x)| \leq x^\lambda$ for all sufficiently large $x > 0$ in $M$.

**Theorem 5.1.** [vdD] *Suppose the dimension of $\mathcal{M}$ is finite. Then the rational rank of the value group $G$ of $\mathcal{M}$ is bounded by* $\dim(\mathcal{M})$.

**Theorem 5.2.** *Let $\mathcal{M} = \langle M, <, +, \cdot, \ldots\rangle$ be a power bounded o-minimal expansion of a real closed field, $v$ its natural valuation, $G$ its value group, $k$ its residue field, $\mathcal{P} \subseteq \mathcal{M}$ its prime model.*

*Then $\mathcal{M}$ is $\aleph_\alpha$-saturated if and only if*

(1) *$(G, +, 0, <)$ is $\aleph_\alpha$-saturated,*
(2) *$k \cong \mathbb{R}$,*
(3) *for every substructure $\mathcal{M}'$ with $\dim(\mathcal{M}'/\mathcal{P}) < \aleph_\alpha$, every pseudo Cauchy sequence in $M'$ has a pseudo limit in $M$.*

*Proof.* We assume conditions (1), (2) and (3) and we show that $\mathcal{M}$ is $\aleph_\alpha$-saturated.

Let $q$ be a complete 1-type over $\mathcal{M}$ with parameters in $A \subset M$, with $|A| < \aleph_\alpha$. Let $\mathcal{M}_0$ be an elementary extension of $\mathcal{M}$ in which $q(x)$ is realized, and $x_0 \in M_0$ such that $\mathcal{M}_0 \models q(x_0)$.

To realize $q$ in $\mathcal{M}$ it is necessary and sufficient to realize the cut that $x_0$ makes in $\mathcal{M}' = \mathrm{dcl}(A) \subseteq \mathcal{M}$

$$q'(x) := \{b \leq x \,;\, b \in M,\, q \vdash b \leq x\} \cup \ \{x \leq c \,;\ c \in M,\, q \vdash x \leq c\}.$$

As we will see in realizing the cut $q'$ instead of type $q$ some care is needed in the case of $\aleph_0$-saturation. If $q'(x)$ contains an equality, the result is obvious. So suppose that in $q'(x)$ there are only strict inequalities.

Set

$$B := \{b \in M' \,;\, q \vdash b < x\} \text{ and } C := \{c \in M' \,;\, q \vdash x < c\}$$

and consider the following subset of $v(M_0)$:

$$\Delta = \{v(d - x_0) \,|\, d \in M'\}.$$

There are three cases to consider:

($a$) *Immediate transcendental case*: $\Delta$ has no largest element.

($b$) *Value transcendental case*: $\Delta$ has a largest element $\gamma \notin v(M')$.

($c$) *Residue transcendental case*: $\Delta$ has a largest element $\gamma \in v(M')$.

($a$) $\Delta$ has no largest element. Then

$$\forall\, d \in M' \;\exists\, d' \in M' : v(d' - x_0) > v(d - x_0).$$

Let $\{v(d_\lambda - x_0)\}_{\lambda<\mu}$ be cofinal in $\Delta$, then $\{d_\lambda\}_{\lambda<\mu}$ is a pseudo Cauchy sequence in $M'$ and $\dim(\mathcal{M}'/P) \leq |A| < \aleph_\alpha$. Condition (3) implies the existence of a pseudolimit $a \in M$ of $\{d_\lambda\}_{\lambda<\mu}$. We claim that $a$ realizes $q'(x)$ in $\mathcal{M}$. The ultrametric inequality gives

$$v(a - x_0) = v(a - d_\lambda + d_\lambda - x_0) \geq \min\{v(a - d_\lambda), v(d_\lambda - x_0)\}.$$

Moreover, from properties of pseudo Cauchy sequences we have

$$v(a - d_\lambda) = v(d_{\lambda+1} - d_\lambda) = v(x_0 - d_\lambda),$$

which implies that for all $\lambda$, $v(a-x_0) \geq v(d_\lambda - x_0)$. Thus for all $d \in \mathcal{M}'$, $v(a - x_0) > v(d - x_0)$. We want to show that $a$ fills the cut determined by $B$ and $C$, and so $a$ realizes $q'$. Let $b \in B$, if $a \leq b$ then $a \leq b < x_0$, and this implies $v(a - x_0) \geq v(b - x_0)$, which is a contradiction. Hence $b < a$. In a similar way we can how that if $c \in C$ then $a < c$.

($b$) $\Delta$ has a largest element $\gamma \notin v(M')$. Fix $d_0 \in M'$ such that $v(d_0 - x_0) = \gamma$ is the maximum of $\Delta$. Assume $d_0 \in B$ (the case $d_0 \in C$ is treated similarly). Let $\Delta_1 = \{v(c - d_0) : c \in C\}$ and $\Delta_2 = \{v(b - d_0) : b \in B, b > d_0\}$.

*Claim.* $\Delta_1 < \gamma < \Delta_2$.

From $d_0 \in B$ it follows $v(c - x_0) < \gamma$ for all $c \in C$. Thus

$$v(c - d_0) = v(c - x_0 + x_0 - d_0) = \min\{v(c - d_0), v(x_0 - d_0)\} =$$
$$v(c - x_0) < \gamma$$

Let $b \in B$ and $b \geq d_0$ then $v(x_0 - b) \geq v(x_0 - d_0) = \gamma$, and by the maximality of $\gamma$ the equality must hold. Thus,

$$v(b - d_0) = v(b - x_0 + x_0 - d_0) \geq \min\{v(b - d_0), v(x_0 - d_0)\} = \gamma.$$

Since $\gamma \notin v(M')$ we have $v(b - d_0) > \gamma$, which completes the proof of the Claim.

Consider the set of formulas

$$t(y) = \{v(c - d_0) < y; c \in C\} \cup \{y < v(b - d_0); b \in B, b > d_0\}.$$

This is a type over $G$ with parameters in $v(M')$. Let $G' = v(M')$. If $\aleph_\alpha > \aleph_0$ then $\mathrm{card}(G') < \aleph_\alpha$ and by hypothesis (1) we can realize $t(y)$ in $G$.

If $\aleph_\alpha = \aleph_0$ then $\mathcal{M}'$ has finite dimension over the prime field $\mathcal{P}$, and Theorem 5.1 implies that the rational rank of $G'$ is bounded by the dimension of $\mathcal{M}'$ over $\mathcal{P}$. So, we can transform the type $t(y)$ in a type $t'(y)$ where the parameters vary over the finite $\mathbb{Q}$-basis of $G'$. Since $G$ is $\aleph_0$-saturated we can realize $t'(y)$ in $G$. Let $a \in M$, $a > 0$ such that $v(a) = g$. We claim that $a + d_0 \in M$ realizes $q'$. From the definition of the type $t(y)$, it follows that for all $c \in C$ and for all $b \in B$ such that $b > d_0$,

$$v(c - d_0) < v(a) < v(b - d_0),$$

and by order property of the valuation $v$ we have that for all $c \in C$ and for all $b \in B$ such that $b > d_0$

$$b - d_0 < a < c - d_0$$

which implies for all $c \in C$ and for all $b \in B$

$$b < a + d_0 < c,$$

hence $a$ realizes the type $q'$ in $\mathcal{M}$.

($c$) $\Delta$ has a largest element $\gamma \in v(M')$. Let $d_0 \in M'$ and $a \in M'$ such that $v(d_0 - x_0) = \gamma = v(a)$ (without loss of generality we may assume $a > 0$).

*Claim.* There exist $b_0 \in B$ and $c_0 \in C$ such that for all $b \in B$ with $b \geq b_0$ and for all $c \in C$ with $c \leq c_0$ we have

$$v(b - d_0) = \gamma = v(a) = v(c - d_0).$$

From $v(d_0 - x_0) = v(a)$ it follows that there exists $n \in \mathbb{N}$ such that $na > |x_0 - d_0| > \frac{a}{n}$. We distinguish the two cases according to $d_0 \in B$ and $d_0 \in C$. Assume $d_0 \in B$, and let $b_0 = d_0 + \frac{a}{n}$ and $c_0 = d_0 + na$. Clearly, $b_0 < x_0$, so $b_0 \in B$, and $x_0 < c_0$, so $c_0 \in C$. Moreover, $v(b_0 - d_0) = v(\frac{a}{n}) = v(a) = v(na) = v(c_0 - d_0)$. If $b \in B$, $b > b_0$ and $c \in C$, $c < c_0$, then the following inequalities hold $d_0 < b_0 < b < c < c_0$. Thus, $v(b - d_0) \leq v(b_0 - d_0) = \gamma = v(c_0 - d_0) \leq v(b - d_0)$. Hence, $\gamma = v(b - d_0)$. Similarly, one shows that $\gamma = v(c_0 - d_0) \leq v(c - d_0) \leq v(b_0 - d_0) = \gamma$, and so $\gamma = v(c - d_0)$.

Assume $d_0 \in C$, and let $b_0 = d_0 - na$ and $c_0 = d_0 - \frac{a}{n}$. Similar calculations show that $v(c - d_0) = \gamma = v(b - d_0)$ for $c \in C$, $c < c_0$, and $b \in B$, $b > b_0$.

Our aim is to show that there is an element $r \in M$ which realizes the cut $q'(x)$. It is enough to show that there is $r'' \in M$ realizing

$$\left\{\frac{b-d_0}{a} < x; b \in B, b \geq b_0\right\} \cup \left\{x < \frac{c-d_0}{a}; c \in C, c \leq c_0\right\}. \tag{3}$$

Indeed, $r' = r''a \in M$ realizes

$$\{b - d_0 < x; b \in B, b \geq b_0\} \cup \{x < c - d_0; c \in C, c \leq c_0\} \tag{4}$$

and so $r = r' + d_0 \in M$ realizes $q'(x)$. Assume $d_0 \in B$. The claim implies that for all $b \in B$, $b \geq b_0$, and for all $c \in C$, $c \leq c_0$ we have

$$v\left(\frac{b-d_0}{a}\right) = v\left(\frac{x_0-d_0}{a}\right) = v\left(\frac{c-d_0}{a}\right) = 0,$$

and taking residues the following inequalities hold in $\mathbb{R}$, the residue field

$$\overline{\frac{b-d_0}{a}} < \overline{\frac{x_0-d_0}{a}} < \overline{\frac{c-d_0}{a}}.$$

(Notice that the inequalities are strict because of the maximality of $v(a)$ in $\Delta$.) The cut in $\mathbb{R}$

$$\left\{\overline{\frac{b-d_0}{a}}; b \in B, b \geq b_0\right\} \cup \left\{\overline{\frac{c-d_0}{a}}; c \in C, c \leq c_0\right\}$$

is realized in $\mathbb{R}$ by $\overline{\frac{x_0-d_0}{a}}$. If $r'' \in M$ is such that $\overline{r''} = \overline{\frac{x_0-d_0}{a}}$ then $r''$ realizes (3) in $\mathcal{M}$. The proof in the case $d_0 \in C$ is similar and we omit it.

We now assume that $\mathcal{M}$ is $\aleph_\alpha$-saturated and we show that conditions (1),(2) and (3) hold.

(1) Let $q(x)$ be a type with set of parameters $A \subset G$ such that $\mathrm{card}(A) < \aleph_\alpha$, e.g. suppose $A = \{g_\mu : \mu < \lambda\}$, where $\lambda < \aleph_\alpha$. We have to show that $q(x)$ is realized in $G$. Without loss of generality we can assume that $q(x)$ is a complete type. Let $H$ be the divisible hull of $A$ in $G$. Notice that $\mathrm{card}(H) < \aleph_\alpha$ for $\aleph_\alpha > \aleph_0$.

It is enough to realize in $G$ the set

$$\{g \leq x\,;\, g \in H, q(x) \vdash g \leq x\} \cup \{x \leq g\,;\, g \in H, q(x) \vdash x \leq g\}.$$

If the set contains an equality, we are done. So suppose that we only have strict inequalities.

For every $\mu \in \lambda$ fix an element $a_\mu \in M$, $a_\mu > 0$, such that $v(a_\mu) = g_\mu$. If $g \in H$ and $g = q_1 g_{i_1} + \cdots + q_m g_{i_m}$ with $q_1, \ldots, q_m \in \mathbb{Q}$, then $g = v(a_{i_1}^{q_1} \cdot \cdots \cdot a_{i_m}^{q_m})$ where for simplicity we choose $a_{i_j}^{q_j} > 0$ for all $j \in \{1, \ldots, m\}$. Let

$$H_1 = \{g \in H; q(x) \vdash g < x\} \text{ and } H_2 = \{g \in H; q(x) \vdash x < g\}$$

and consider

$$q'(x) = \{ka_{i_1}^{q_1} \cdot \dots \cdot a_{i_k}^{q_k} < x; k \in \mathbb{N}, v(a_{i_1}^{q_1} \cdot \dots \cdot a_{i_k}^{q_k}) \in H_2\} \cup$$
$$\{kx < a_{i_1}^{q_1} \cdot \dots \cdot a_{i_k}^{q_k}; k \in \mathbb{N}, v(a_{i_1}^{q_1} \cdot \dots \cdot a_{i_k}^{q_k}) \in H_1\}.$$

Since $\mathcal{M}$ is a dense linear ordering without endpoints, $q'(x)$ is finitely realizable in $\mathcal{M}$. Thus $q'(x)$ is a type in the parameters $\{a_\mu\}_{\mu<\lambda}$.

Since $\mathcal{M}$ is $\aleph_\alpha$-saturated it follows that $q'(x)$ is realized in $\mathcal{M}$, say by $a$. Then $v(a)$ realizes $q(x)$.

(2) Since $(M, +, 0, <)$ is $\aleph_\alpha$-saturated Theorem 3.5 implies that all Archimedean components are isomorphic to $\mathbb{R}$, but there is only one Archimedean component and this is the residue field, so $k \cong \mathbb{R}$.

(3) Let $(a_\nu)_{\nu<\mu}$ be a pseudo Cauchy sequence in $\mathcal{M}'$, where $\mathcal{M}'$ is a substructure of $\mathcal{M}$ and $\dim(\mathcal{M}'/\mathcal{P}) = \lambda < \aleph_\alpha$. Let $\{b_\alpha; \alpha < \lambda\}$ be a basis of $\mathcal{M}'$ over the prime field $\mathcal{P}$. Then all elements $a_\nu$ are definable in terms of finitely many elements of the basis with coefficients in the prime field $\mathcal{P}$. Recall that the prime field $\mathcal{P}$ coincides with $\mathrm{dcl}(\emptyset)$ hence every element of $\mathcal{P}$ is definable by a formula without paramenters. This is crucial in the case of $\aleph_0$-saturation. Let

$$q_1(x) = \{n|x - a_{\nu+1}| < |a_\nu - a_{\nu+1}|; \nu < \mu, n \in \mathbb{N}\}.$$

Then $q_1(x)$ is a set of formulas in $\lambda$ parameters (in the case of $\aleph_0$-saturation the parameters are only finitely many). Moreover, $q_1(x)$ is finitely satisfied in $\mathcal{M}$ since $(a_\mu)_{\mu<\lambda}$ is pseudo Cauchy. Hence $q_1(x)$ is a type, and a realization of $q_1(x)$ in $\mathcal{M}$ (which is $\aleph_\alpha$-saturated) is a pseudo limit of the sequence. □

## 6. $\aleph_\alpha$-saturated o-minimal expansions

If we take any o-minimal expansion of a real closed field (not necessarily power bounded) we obtain the following analogue of Theorem 4.1.

**Theorem 6.1.** *Let $\mathcal{M} = \langle M, <, +, \cdot, \dots \rangle$ be an o-minimal expansion of a real closed field, $v$ its natural valuation, $G$ its value group, $k$ its residue field, $\mathcal{P} \subset \mathcal{M}$ its prime model.*

*Then $\mathcal{M}$ is $\aleph_\alpha$-saturated $\iff$ for every substructure $\mathcal{M}' \subset \mathcal{M}$ such that $\dim(\mathcal{M}'/\mathcal{P}) < \aleph_\alpha$, then*

1. *$(G, <, +, v(\mathcal{M}'))$ is $\aleph_\alpha$-saturated,*
2. *$k \cong \mathbb{R}$,*
3. *every pseudo Cauchy sequence in $\mathcal{M}'$ has a pseudo limit in $\mathcal{M}$.*

The proof is analogous to that of Theorem 5.2, and we omit it. We just point out that in the value transcendental case the expansion $(G, <, +, v(\mathcal{M}'))$ of the value group is needed for $\aleph_0$-saturation. In the

power bounded case the valuation inequality allows us to get rid of the parameters in $v(\mathcal{M}')$. By Miller's dichotomy (see [M2]) the exponential function is definable if we are not in the power bounded case. In a forthcoming paper we further analyze Theorem 6.1 in that particular case. Finally, note that if in Theorem 5.2 we assume $\mathcal{M}$ is just a real closed field, then we obtain exactly Theorem 4.1: the prime model $\mathcal{P}$ is the field of real algebraic numbers, and $\mathcal{M}'$ is a submodel of finite dimension over $\mathcal{P}$ if and only if it is of finite absolute transcendence degree.

## References


[AK94] N. L. Alling and S. Kuhlmann, On $\eta_\alpha$-Groups and Fields, *Order*, **11** (1994), pp. 85–92.

[DKS10] P. D'Aquino, J.F. Knight and S. Starchenko, Real closed fields and models of Peano arithmetic, *J. Symb. Logic*, **75(1)** (2010), pp. 1–11.

[vdD] L. van den Dries, T-Convexity and Tame Extensions II, *J. Symb. Logic*, **62** (1997), pp. 14–34.

[KKMZ02] F.-V. Kuhlmann, S. Kuhlmann, M. Marshall, M. Zekavat, Embedding ordered fields in formal power series fields, *J. Pure Appl. Algebra*, **169** (2002), pp. 71–90.

[Ku90] S. Kuhlmann, Groupes abéliens divisibles ordonnés, *Séminaire sur les Structures Algébriques Ordonnées, Sélection d'exposés 1984-1987*, **Vol.1** (1990), pp. 3–14.

[Ku01] S. Kuhlmann, **Ordered Exponential Fields**, The Fields Institute Monograph Series, vol 12. Amer. Math. Soc. 2000.

[M1] C. Miller, Exponentiation is hard to avoid, *Proceedings of the American Mathematical Society*, **vol. 122** (1994), pp. 257-259.

[M2] C. Miller, A growth dichotomy for o-minimal expansions of ordered fields, *Logic: from foundations to applications*, **European Logic Colloquium 1993, (eds. W.Hodges et al.)** (Oxford University Press 1993), pp. 385-399.

[PS] A. Pillay and C. Steinhorn, Definable sets in ordered structures,I, *Proceedings of the American Mathematical Society*, **vol. 295** (1986), pp. 565-592.

[P] A. Prestel, **Valued Fields**, Springer, 2005.

[R] J.G. Rosenstein, **Linear Orderings**, Academic Press, 1982.

[W] A. Wilkie, Model completeness results for expansions of the ordered field of real numbers by restricted pfaffian functions and the exponetial functions, *Journal Amer. Math. Soc.*, **9** (1996), pp. 1051-1094.

FB Mathematik & Statistik, Universität Konstanz, Germany
*E-mail address*: annalisa.conversano@uni-konstanz.de

Dipartimento di Matematica, Seconda Università di Napoli, Italy
*E-mail address*: paola.daquino@unina2.it

FB Mathematik & Statistik, Universität Konstanz, Germany
*E-mail address*: salma.kuhlmann@uni-konstanz.de